

# **STATISTICAL TESTING FOR ASYMPTOTIC NO-ARBITRAGE IN FINANCIAL MARKETS**

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## **Abstract**

This paper deals with the notion of a large financial market and the concept of asymptotic no-arbitrage. This concept is closely related to that of contiguity of the equivalent martingale measures. Here, assuming a time varying ARCH return for financial asset, we derive the stochastic expansion of the log-likelihood ratio for the equivalent martingale measure. Then we give a sufficient condition that, there is no asymptotic arbitrage. Related to this condition, a test statistic for this is proposed. The asymptotics are elucidated. Numerical studies of the test are provided, and they show that our test is useful for testing asymptotic no-arbitrage.

## **1. Introduction**

We introduce a large financial market as a sequence of ordinary security market models in discrete time. The large financial market is

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described by a sequence of standard general models of continuous trading. According to Kabanov and Kramkov [10], paying attention to such market is connected to detect the absence of asymptotic arbitrage. In any economic equilibrium, it should not be possible to purchase at zero cost a bundle of goods that will strictly increase some agent's utility. For as long as such an 'arbitrage opportunity' exists, the agent in question will purchase the bundle, and one will continue to do so until either its price rise or it ceases to increase one's utility. The absence of arbitrage opportunities is thus a necessary condition for an economic equilibrium. In the literature of economics, most of them have considered many strategies under the absence of arbitrage opportunities (e.g., Gurupdesh [6]). However, we investigate whether there exist the arbitrage opportunities because this investigation is more important than the consideration of the above strategies. It is shown that, this property is closely related to the contiguity of the equivalent martingale measures. As one of the crucial roles in the theory of asymptotic arbitrage, there is a concept of contiguity of probability measure. We discuss the problem of statistical testing for asymptotic no-arbitrage in financial markets in terms of the asymptotic properties of the likelihood ratio and in terms of contiguity. In this paper, denoting by  $X_{t,N}$  the value of a financial return at time  $t$  with observation length  $N$ , we assume that  $\{X_{t,N}\}$  is generated by a time varying ARCH (tv ARCH) model with mean  $\mu_{t,N}$ , volatility  $\sigma_{t,N}$ , and innovation density  $\phi(\cdot)$ . Then, the asymptotics of the log-likelihood ratio  $\Lambda_N$  between the original probability measure  $P_N$  and an equivalent martingale measure  $\tilde{P}_N$  are investigated. Based on them, we give a sufficient condition that there is no asymptotic arbitrage. Next, a statistic for testing no asymptotic arbitrage is proposed. Dahlhaus and Rao [3] developed a systematic asymptotic estimation theory for tv ARCH models. Thanks to their results, we can show that the test statistic converges in probability to a quantity, which describes the sufficient condition. Numerical studies for testing no asymptotic arbitrage are given, and they illuminate some interesting feature of our problem.

Concretely, this paper is organized as follows. Section 2 gives the stochastic expansion of  $\Lambda_N$  under  $P_N$ , which leads to a sufficient condition that there is no asymptotic arbitrage in our financial market. The sufficient condition is described by a fundamental quantity  $A_{t,N}$ , which depends on  $\mu_{t,N}$ ,  $\sigma_{t,N}$  and  $\phi(\cdot)$ . For concrete distribution forms of  $\phi(\cdot)$ , we can write the no asymptotic arbitrage condition in terms of  $\mu_{t,N}$  and  $\sigma_{t,N}$ . In Section 3, we propose a consistent estimator  $\hat{A}_{t,N}$  of  $A_{t,N}$  by use of Dahlhaus and Rao's results [3]. For a tv ARCH model, Section 4 provides numerical studies for  $\hat{A}_{t,N}$ , and shows that our test is useful in the testing problem. The proofs of the results are relegated to Section 5.

## 2. Asymptotic No-arbitrage

First, we introduce a financial asset model. Let  $S_{t,N}$ , ( $t = 1, \dots, N$ ,  $N \in \mathbf{N}$ ), be generated by

$$S_{t,N} = S_{0,N} \exp \{X_{1,N} + \dots + X_{t,N}\}, \quad X_{t,N} = \mu_{t,N} + \sigma_{t,N} \epsilon_t, \quad (2.1)$$

where

$$\mu_{t,N} = \mu \left( \frac{t}{N} \right), \quad (2.2)$$

$$\sigma_{t,N}^2 = a_0 \left( \frac{t}{N} \right) + \sum_{j=1}^p a_j \left( \frac{t}{N} \right) (X_{t-j,N} - \mu_{t-j,N})^2, \quad (2.3)$$

and  $\epsilon_t$ 's are independent and identically distributed (0, 1) with p.d.f.  $\phi(\cdot)$ .  $S_{t,N}$  is supposed to be the value of financial asset at time  $t$  with observed stretch  $N$ . Let  $\mathcal{F}_{t,N}$  be the  $\sigma$ -algebra generated by  $(X_{1,N}, \dots, X_{t,N})$ , and assume that  $\mu_{t,N}, \sigma_{t,N}^2 \in \mathcal{F}_{t-1,N}$ .

We set down the following assumptions.

**Assumption 1.** The stochastic process  $\{X_{t,N} : t = 1, \dots, N\}$  has a time varying ARCH(p) (tv ARCH(p)) representation defined in (2.1), where the parameters satisfy the following properties. There exist constants  $0 < \rho, Q, M < \infty, 0 < \nu < 1$ , and a sequence  $\{l(j)\}$  of positive numbers such that  $\inf_u a_0(u) > \rho$  and

$$\sup_u a_j(u) \leq \frac{Q}{l(j)},$$

$$Q \sum_{j=1}^{\infty} \frac{1}{l(j)} \leq (1 - \nu),$$

$$|a_j(u) - a_j(v)| \leq M \frac{|u - v|}{l(j)},$$

where  $\{l(j)\}$  satisfies

$$\sum_{j \geq 1} \frac{j}{l(j)} < \infty.$$

An example of such a sequence  $\{l(j)\}$  is

$$l(j) = \begin{cases} 1 & j = 1, \\ j^2 \log^{1+\kappa} j & j > 1, \end{cases}$$

with some  $\kappa > 0$  or

$$l(j) = \eta^j,$$

for some  $\eta > 1$ . Condition  $Q \sum_{j=1}^{\infty} \frac{1}{l(j)} \leq (1 - \nu)$  implies that  $E(\sigma_{t,N}^2)$  is uniformly bounded over  $t$  and  $N$ .

**Assumption 2.** (i) For some  $\delta > 0$ ,  $E(|\epsilon_t|^{4(1+\delta)}) < \infty$ .

(ii) There exists an interval  $I \ni 0$  such that  $\phi(x) > 0$  on  $I$ .

Let  $\Omega$  be a compact set defined by

$$\Omega = \left\{ \boldsymbol{\theta} = (\boldsymbol{\mu}, \boldsymbol{\alpha})', \boldsymbol{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_p) : \sum_{j=1}^p \alpha_j \leq 1, \right. \\ \left. \rho_1 \leq \alpha_0 \leq \rho_2, \rho_1 \leq \alpha_i, \text{ for } i = 1, \dots, p, \boldsymbol{\mu} \in \mathbf{R} \right\},$$

where  $0 < \rho_1 < \rho_2 < \infty$ .

**Assumption 3.** (i)  $\boldsymbol{\theta}_u \equiv (\boldsymbol{\mu}(u), \alpha_0(u), \alpha_1(u), \dots, \alpha_p(u))' \in \Omega$  for each  $u \in [0, 1]$ .

(ii)  $\boldsymbol{\mu}(\cdot)$  and  $\alpha_j(\cdot)$ 's are continuously three times differentiable, and there exists a positive constant  $C$  such that

$$\sup_u \left| \frac{\partial^i \alpha_j(u)}{\partial u^i} \right| \leq C, \quad \sup_u \left| \frac{\partial^i \boldsymbol{\mu}(u)}{\partial u^i} \right| \leq C,$$

for  $i = 1, 2, 3$  and  $j = 0, 1, \dots, p$ .

Let  $P_N$  be the probability distribution of  $(X_{1,N}, \dots, X_{N,N})$  and  $\tilde{P}_N$  be a probability distribution satisfying

$$\tilde{E}\{S_{t,N} | \mathcal{F}_{t-1,N}\} = S_{t-1,N} \quad (\tilde{P}_N \text{-a.s.}), \quad (2.4)$$

where  $\tilde{E}(\cdot)$  is the expectation with respect to  $\tilde{P}_N$ . Write  $\tilde{\phi}(\sigma_{t,N}) = \int_{-\infty}^{\infty} \exp\{\mathfrak{G}_{t,N}x\} \phi(x) dx$ .

If  $\mu_{t,N} = -\log \tilde{\phi}(\sigma_{t,N})$  under  $\tilde{P}_N$ , then  $S_{t,N}$  is a martingale satisfying (2.4). In fact,

$$\begin{aligned} \tilde{E}\{S_{t,N} | \mathcal{F}_{t-1,N}\} &= S_{t-1,N} \tilde{E}\{e^{X_{t,N}} | \mathcal{F}_{t-1,N}\} \\ &= S_{t-1,N} e^{\mu_{t,N}} \tilde{E}\{e^{\sigma_{t,N} \epsilon_t} | \mathcal{F}_{t-1,N}\} \\ &= S_{t-1,N} e^{\mu_{t,N}} \tilde{\phi}(\sigma_{t,N}) \\ &= S_{t-1,N} e^{\mu_{t,N} + \log \tilde{\phi}(\sigma_{t,N})} \end{aligned} \quad (2.5)$$

$$= S_{t-1, N}.$$

Define the likelihood ratio  $Z_N$  between  $P_N$  and  $\tilde{P}_N$  by

$$Z_N \equiv \frac{d\tilde{P}_N}{dP_N} = \prod_{t=1}^N \frac{\frac{1}{\sigma_{t, N}} \phi \left\{ \frac{X_{t, N} + \log \tilde{\phi}(\sigma_{t, N})}{\sigma_{t, N}} \right\}}{\frac{1}{\sigma_{t, N}} \phi \left\{ \frac{X_{t, N} - \mu_{t, N}}{\sigma_{t, N}} \right\}} = \prod_{t=1}^N \frac{\phi\{\epsilon_t + A_{t, N}\}}{\phi\{\epsilon_t\}}, \quad (2.6)$$

where

$$A_{t, N} = \frac{\mu_{t, N} + \log \tilde{\phi}(\sigma_{t, N})}{\sigma_{t, N}}.$$

The log-likelihood ratio  $\Lambda_N$  has the following representation.

$$\Lambda_N = \log Z_N = \sum_{t=1}^N \{\log \phi(\epsilon_t + A_{t, N}) - \log \phi(\epsilon_t)\}. \quad (2.7)$$

Further, we make the following assumption on  $\phi(\cdot)$ .

**Assumption 4.** (i)  $\phi(\cdot)$  is continuously three times differentiable on  $\mathbf{R}$ .

(ii)  $D = \frac{d}{dx}$  and  $\int$  are interchangeable.

(iii)  $\int |D\phi(x)| dx < \infty$ ,  $\int |D^2\phi(x)| dx < \infty$ .

(iv)  $\mathcal{F}(\epsilon) \equiv \int \left\{ \frac{D\phi(x)}{\phi(x)} \right\}^2 dx < \infty$ .

Then, we have the following result.

**Theorem 1.** *Suppose that Assumptions 1-4 hold. If  $E|A_{t, N}|^2 = O\left(\frac{1}{t \log N}\right)$ , then the log-likelihood ratio  $\Lambda_N$  has the following stochastic expansion under  $P_N$ ;*

$$\Lambda_N = S - \frac{1}{2} \mathcal{F}(\epsilon) \sum_{t=1}^N A_{t,N}^2 + o_p(1), \quad (2.8)$$

where  $S$  is a random variable satisfying  $-\infty < S < \infty$  a.s., and  $\sum_{t=1}^N A_{t,N}^2 < \infty$  a.s.

The proofs of the theorems are relegated to Section 5. From the above, we have the following result.

**Corollary 1.** *Under the same assumptions as in Theorem 1,*

$$P\left\{ \lim_{N \rightarrow \infty} Z_N > 0 \right\} = 1.$$

From Corollary 1, Theorem 1 of Shiryaev ([11], Chapter VI, Theorem 1, p. 560) implies that, for the financial asset  $\{S_{t,N}\}$ , there is no asymptotic arbitrage.

Next, assuming a special form of  $\phi(\cdot)$ , we consider conditions for  $\mu_{t,N}$ ,  $\sigma_{t,N}^2$  to satisfy the assumption  $E|A_{t,N}|^2 = O\left(\frac{1}{t \log N}\right)$ .

**Corollary 2.** *Assume that  $\epsilon_t$ 's are i.i.d. symmetric  $\alpha$ -stable r.v.'s for  $1 < \alpha \leq 2$ . This means that, the moment generation function (not characteristic function) of  $\epsilon_1$  is given by*

$$\tilde{\phi}(t) = E(e^{t\epsilon_1}) = e^{-\sigma_0 |t|^\alpha}, \quad t \in \mathbf{R}, \quad (2.9)$$

with a scaling factor  $\sigma_0 > 0$ . In this case, if

$$\left| \frac{\mu_{t,N}}{\sigma_{t,N}} - \sigma_0 \sigma_{t,N}^{\alpha-1} \right| \leq C \frac{1}{\sqrt{t \log N}}, \quad \text{a.s.}, \quad (2.10)$$

for some  $C > 0$ , and  $t = 1, \dots, N$ , then  $E|A_{t,N}|^2 = O\left(\frac{1}{t \log N}\right)$  holds.

It is known that, if  $\alpha = 2$ ,  $\{\epsilon_t\}$  follows Gaussian distribution, if  $\alpha = 1$ ,  $\{\epsilon_t\}$  follows Cauchy distribution, which has infinite mean.

### 3. Estimation and Test

In this section, we consider estimation of  $A_{t,N}$ . We first define a segment (kernel) estimator of  $\boldsymbol{\theta}_{u_0} = (\mu(u_0), \alpha_0(u_0), \alpha_1(u_0), \dots, \alpha_p(u_0))'$  for  $u_0 \in (0, 1]$ . Let  $t_0 \in N$  such that  $|u_0 - t_0 / N| < 1 / N$ . The estimator considered in this section is the minimizer of the weighted quasi-likelihood

$$\mathcal{L}_{t_0, N}(\boldsymbol{\theta}) := \sum_{k=1}^N \frac{1}{bN} W\left(\frac{t_0 - k}{bN}\right) l_{k, N}(\boldsymbol{\theta}), \quad (3.1)$$

where

$$l_{k, N}(\boldsymbol{\theta}) = \frac{1}{2} \left( \log w_{k, N}(\boldsymbol{\theta}) + \frac{(X_{k, N} - \mu)^2}{w_{k, N}(\boldsymbol{\theta})} \right),$$

with  $w_{k, N}(\boldsymbol{\theta}) = \alpha_0 + \sum_{j=1}^p \alpha_j (X_{k-j, N} - \mu)^2$ ,

and  $W : \left[-\frac{1}{2}, \frac{1}{2}\right] \rightarrow \mathbf{R}$  is a kernel function of bounded variation with

$\int_{-\frac{1}{2}}^{\frac{1}{2}} W(x) dx = 1$  and  $\int_{-\frac{1}{2}}^{\frac{1}{2}} xW(x) dx = 0$ . That is, we consider,

$$\hat{\boldsymbol{\theta}}_{t_0, N} = \arg_{\boldsymbol{\theta} \in \Omega} \min \mathcal{L}_{t_0, N}(\boldsymbol{\theta}). \quad (3.2)$$

Then, we have the following result.

**Theorem 2.** *Suppose that  $\{X_{t, N} : t = 1, \dots, N\}$  is generated by the tv ARCH(p) process, which satisfies Assumptions 1-2, and the estimator  $\hat{\boldsymbol{\theta}}_{t_0, N} = (\hat{\mu}_{t_0, N}, \hat{\boldsymbol{\alpha}}_{t_0, N})'$  is defined by (3.2). Then, if  $|u_0 - t_0 / N| < 1 / N$ , we have*

$$\hat{A}_{t_0, N} \equiv \frac{\hat{\mu}_{t_0, N} + \log(\tilde{\phi}(w_{t_0, N}(\hat{\boldsymbol{\theta}}_{t_0, N})^{\frac{1}{2}}))}{w_{t_0, N}(\hat{\boldsymbol{\theta}}_{t_0, N})^{\frac{1}{2}}} \xrightarrow{p} \frac{\mu(u_0) + \log \tilde{\phi}(\sigma_{t_0}(u_0))}{\sigma_{t_0}(u_0)} \equiv A_{t_0, N}.$$



Let us give an example of asymptotic no-arbitrage test.

Consider the testing problem

$H_0$  : there is no asymptotic arbitrage

v.s.

$H_1$  : there is asymptotic arbitrage.

Under  $H_0$ , we assume that  $\{\epsilon_t\}$  follows  $\alpha$ -stable distribution defined by (2.9) and

$$\mu_{t,N} = \sigma_0 \sigma_{t,N}^\alpha + \frac{C}{\sqrt{t \log N}} \sigma_{t,N}. \quad (3.3)$$

Then,  $\{X_{t,N}\}$  satisfies the condition (2.10), which implies that there is no asymptotic arbitrage from Corollary 1. In this case, from Dahlhaus and Rao [3], it is seen that the estimator  $\hat{A}_{t_0,N}$  defined in Theorem 2 has the asymptotic normality

$$\sqrt{N}(\hat{A}_{t_0,N} - A_{t_0,N}) \xrightarrow{d} N(0, \Omega).$$

Thus,

$$(i) \hat{A}_{t_0,N} \approx A_{t_0,N} + \frac{1}{\sqrt{N}} N(0, \Omega),$$

$$(ii) A_{t_0,N} = O\left(\frac{1}{\sqrt{t_0 \log N}}\right) \text{ a.s.},$$

which implies that, if  $t_0 = o(N / \log N)$ ,

$$\sqrt{t_0 \log N} \hat{A}_{t_0,N} = O(1) \text{ a.s.}$$

In view of this, if we plot  $\sqrt{t_0 \log N} \hat{A}_{t_0,N}$ , we can detect the existence of asymptotic arbitrage (see Figures 1 and 2). In practice,  $\Omega$  is often unknown, but we may ignore  $Z \sim \frac{1}{\sqrt{N}} N(0, \Omega)$  because  $Z$  converges

faster than  $A_{t_0, N}$  under  $t_0 = o(N / \log N)$ . In addition, it is easy to see that, the test statistic  $\sqrt{t_0 \log N} \hat{A}_{t_0, N}$  diverges under alternative hypothesis.

#### 4. Simulation

In this section, we examine our approach numerically to verify the result of Corollary 2. Similar to Corollary 2,  $\{\epsilon_t\}$  is supposed to be a sequence of  $\alpha$ -stable r.v.'s. Under the condition, we plot  $\sqrt{t \log N} \hat{A}_{t_0, N}$ .

**Example 1.** Let the process  $\{X_{t, N}\}$  be the tv ARCH(1) model defined by,

$$X_{t, N} = \mu_{t, N} + \sigma_{t, N} \epsilon_t,$$

$$\text{where } \mu_{t, N} = \mu\left(\frac{t}{N}\right), \quad \sigma_{t, N}^2 = \alpha_0\left(\frac{t}{N}\right) + \alpha_1\left(\frac{t}{N}\right)(X_{t-1, N} - \mu_{t-1, N})^2,$$

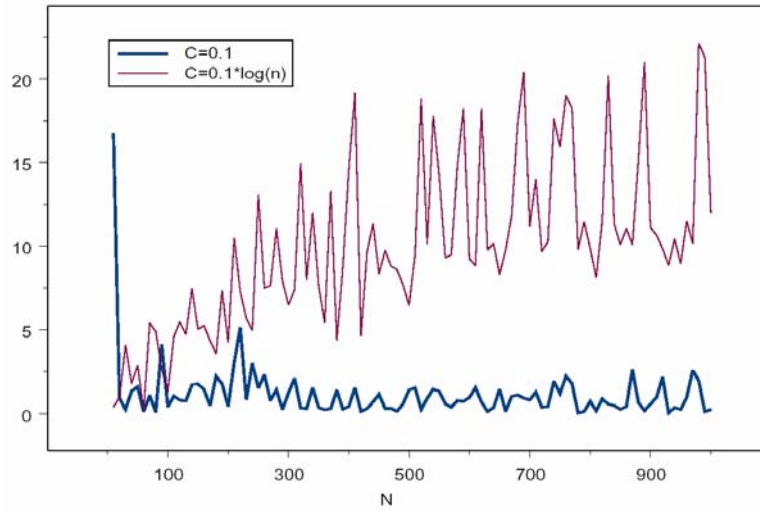
and  $\epsilon_t$ 's are i.i.d. symmetric  $\alpha$ -stable r.v.'s defined by (2.9). In order to satisfy condition (2.10), we assume that,

$$\mu_{t, N} = \sigma_0 \sigma_{t, N}^\alpha + \frac{C}{\sqrt{t \log N}} \sigma_{t, N}, \quad (4.1)$$

$$\sigma_{t, N}^2 = \left(\frac{t}{N}\right) \{\alpha_0 + \alpha_1 (X_{t-1, N} - \mu_{t-1, N})^2\}. \quad (4.2)$$

Given  $\{X_{t, N}\} = (X_{1, N}, \dots, X_{N, N})$ , a segment (kernel) estimator  $\{\hat{\theta}_{t_0, N}\} = (\hat{\theta}_{1, N}, \dots, \hat{\theta}_{N, N})$  is obtained by (3.2). Finally, from  $\{\hat{\theta}_{t_0, N}\}$ , we obtain the estimator  $\{\hat{A}_{t_0, N}\} = (\hat{A}_{1, N}, \dots, \hat{A}_{N, N})$  of  $\{A_{t_0, N}\}$ .

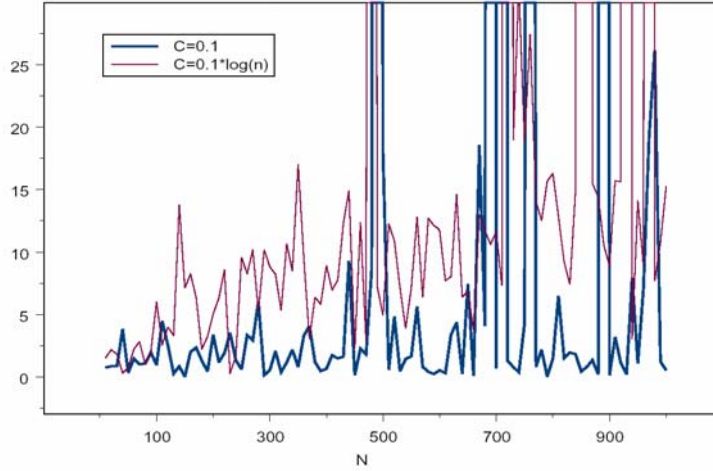
**Model 1.** In Figure 1, we plot the graph of  $\sqrt{t \log N} \hat{A}_{t_0, N}$  for  $\alpha = 2$  (Gaussian distribution),  $N = 10(10)1000$ ,  $\alpha_0 = 0.5$ ,  $\alpha_1 = 0.2$ ,  $W(x) = 6\left(\frac{1}{4} - x^2\right)$ ,  $t_0 = \log N$ , and  $C = 0.1, 0.1 \times \log N$ .



**Figure 1.**  $\alpha = 2.0$ .

From (2.10),  $C$  should be a constant value, which implies that the null hypothesis should be rejected in the case of  $C = 0.1 \times \log N$ . Indeed, the test statistic  $\sqrt{t \log N} \hat{A}_{t_0, N}$  tends to increase as  $N$  increases. On the other hand, in the case of  $C = 0.1$ ,  $\sqrt{t \log N} \hat{A}_{t_0, N}$  converges to a constant value as  $N$  increases.

**Model 2.** In Figure 2, we plot the graph of  $\sqrt{t \log N} \hat{A}_{t_0, N}$  under Model 1 with  $\alpha = 1.5$ .



**Figure 2.**  $\alpha = 1.5$ .

Similarly to Figure 1, in the case of  $C = 0.1$ , the test statistic  $\sqrt{t \log N} \hat{A}_{t_0, N}$  converges to a constant value, and in the case of  $C = 0.1 \times \log N$  tends to increase as  $N$  increases. However, because of heavy tail of the innovation distribution, the fluctuation of  $\sqrt{t \log N} \hat{A}_{t_0, N}$  is larger than that of Figure 1.

In practice, we need to define the rejection region of the test above. For example, by use of bootstrap, we can construct the empirical distribution of  $\sqrt{t \log N} \hat{A}_{t_0, N}$ , then the rejection region can be constructed based on this empirical distribution.

## 5. Proofs

In this section, we give the proofs of results in the previous sections.

**Proof of Theorem 1.** From Corollary 5.1.5 of Fuller [4], it is seen that Taylor expansion of  $\log \phi(\epsilon_t + x)$  at  $x = 0$  leads to,

$$\log \phi(\epsilon_t + A_{t, N}) = \log \phi(\epsilon_t) + \frac{D\phi(\epsilon_t)}{\phi(\epsilon_t)} A_{t, N} + \frac{1}{2} \frac{D^2\phi(\epsilon_t)\phi(\epsilon_t) - \{D\phi(\epsilon_t)\}^2}{\phi(\epsilon_t)^2}$$

$$\times A_{t,N}^2 + O_p((t \log N)^{-\frac{3}{2}}). \quad (5.1)$$

From (5.1), we obtain

$$\Lambda_N = \sum_{t=1}^N \frac{D\phi(\epsilon_t)}{\phi(\epsilon_t)} A_{t,N} + \frac{1}{2} \sum_{t=1}^N \frac{D^2\phi(\epsilon_t)\phi(\epsilon_t) - \{D\phi(\epsilon_t)\}^2}{\phi(\epsilon_t)^2} A_{t,N}^2 + \sum_{t=1}^N O_p((t \log N)^{-\frac{3}{2}}),$$

which implies

$$\Lambda_N = \sum_{t=1}^N \frac{D\phi(\epsilon_t)}{\phi(\epsilon_t)} A_{t,N} + \frac{1}{2} \sum_{t=1}^N \frac{D^2\phi(\epsilon_t)\phi(\epsilon_t) - \{D\phi(\epsilon_t)\}^2}{\phi(\epsilon_t)^2} A_{t,N}^2 + o_p(1). \quad (5.2)$$

Let

$$Y_{t,N} = \frac{D\phi(\epsilon_t)}{\phi(\epsilon_t)} A_{t,N} + \frac{1}{2} \left[ \frac{D^2\phi(\epsilon_t)\phi(\epsilon_t) - \{D\phi(\epsilon_t)\}^2}{\phi(\epsilon_t)^2} + \mathcal{F}(\epsilon) \right] A_{t,N}^2.$$

Then,

$$\begin{aligned} E[Y_{t,N} | \mathcal{F}_{t-1}] &= E \left\{ \frac{D\phi(\epsilon_t)}{\phi(\epsilon_t)} \right\} A_{t,N} \\ &\quad + \frac{1}{2} \left[ E \left\{ \frac{D^2\phi(\epsilon_t)}{\phi(\epsilon_t)} \right\} - E \left\{ \frac{D\phi(\epsilon_t)}{\phi(\epsilon_t)} \right\}^2 + \mathcal{F}(\epsilon) \right] A_{t,N}^2 = 0, \end{aligned} \quad (5.3)$$

because  $E \left\{ \frac{D^i\phi(\epsilon_t)}{\phi(\epsilon_t)} \right\} = \int D^i\phi(x) dx = 0$ ,  $i = 1, 2$  (by Assumption 3(ii)),

and  $E \left\{ \frac{D\phi(\epsilon_t)}{\phi(\epsilon_t)} \right\}^2 = \mathcal{F}(\epsilon)$ .

Also, we can see that

$$\begin{aligned} E \left| \sum_{t=1}^N Y_{t,N} \right| &\leq \sum_{t=1}^N E |Y_{t,N}| \\ &\leq \sum_{t=1}^N E \left| \frac{D\phi(\epsilon_t)}{\phi(\epsilon_t)} A_{t,N} \right| + \frac{1}{2} \sum_{t=1}^N E \left| \left\{ \frac{D^2\phi(\epsilon_t)}{\phi(\epsilon_t)} - \left( \frac{D\phi(\epsilon_t)}{\phi(\epsilon_t)} \right) + \mathcal{F}(\epsilon) \right\} A_{t,N}^2 \right| \end{aligned}$$

$$\leq \sum_{t=1}^N E \left| \frac{D\phi(\epsilon_t)}{\phi(\epsilon_t)} \right| E|A_{t,N}| + \frac{1}{2} \sum_{t=1}^N \left\{ E \left| \frac{D^2\phi(\epsilon_t)}{\phi(\epsilon_t)} \right| + E \left| \frac{D\phi(\epsilon_t)}{\phi(\epsilon_t)} \right|^2 + \mathcal{F}(\epsilon) \right\} E|A_{t,N}^2|$$

(5.4)

<  $\infty$ . (by Assumption 4(iii) and (iv))

Hence,  $\{Y_{t,N}\}$  is a martingale difference sequence. By Doob's martingale convergence theorem (Hall-Heyde [7], Theorem 1.3.9), there exists a random variable  $S$  such that  $E|S| < \infty$ , and

$$\sum_{t=1}^N Y_{t,N} \rightarrow S, \quad \text{a.s.} \quad (5.5)$$

Since,  $\sum_{t=1}^N E|A_{t,N}^2| < \infty$ ,  $A_{t,N}^2 \geq 0$  a.s., and  $E|S| < \infty$ , we observe that,

$$\sum_{t=1}^N A_{t,N}^2 < \infty, \quad \text{a.s., and } -\infty < S < \infty, \quad \text{a.s.} \quad (5.6)$$

From (5.2), (5.5), and (5.6), it follows that

$$\Lambda_N = \sum_{t=1}^N Y_{t,N} - \frac{1}{2} \mathcal{F}(\epsilon) \sum_{t=1}^N A_{t,N}^2 + o_p(1) = S - \frac{1}{2} \mathcal{F}(\epsilon) \sum_{t=1}^N A_{t,N}^2 + o_p(1),$$

which together with (5.6) implies the conclusion (2.8).  $\square$

**Proof of Corollary 1.** Recall  $Z_N = \exp \Lambda_N$ , and by Theorem 1,  $\Lambda_N > -\infty$  a.s., i.e.,  $P\{\lim_{N \rightarrow \infty} \Lambda_N > -\infty\} = 1$ . Therefore,

$$P\{\lim_{N \rightarrow \infty} Z_N > 0\} = P\{\lim_{N \rightarrow \infty} e^{\Lambda_N} > e^{-\infty}\} = 1. \quad \square$$

**Proof of Theorem 2.** First, by Theorem 2 of Dahlhaus and Rao [3], we can see

$$\hat{\boldsymbol{\theta}}_{t_0, N} \xrightarrow{p} \boldsymbol{\theta}_{u_0}, \quad (5.7)$$

where  $\hat{\boldsymbol{\theta}}_{t_0, N} = \left( \hat{\mu}\left(\frac{t_0}{N}\right), \hat{a}_0\left(\frac{t_0}{N}\right), \hat{a}_1\left(\frac{t_0}{N}\right), \dots, \hat{a}_p\left(\frac{t_0}{N}\right) \right)$ .

Now,  $\{\theta_{u_0}\}$  is a parameter of ARCH process  $\{\tilde{X}_t(u_0)\}$ . For each given  $u_0 \in (0, 1]$ , the stochastic process  $\{\tilde{X}_t(u_0)\}$  is a stationary ARCH process associated with the tv ARCH(p) process at times point  $u_0$ . Similar to Theorem 1 of Dahlhaus and Rao [3], we can show

$$X_{t_0, N}^2 \xrightarrow{p} \tilde{X}_{t_0}(u_0)^2, \quad (5.8)$$

where  $\tilde{X}_t(u_0) = \mu(u_0) + \sigma_t(u_0)\epsilon_t$ , with  $\sigma_t(u_0)^2 = \alpha_0(u_0) + \sum_{j=1}^{\infty} \alpha_j(u_0) (\tilde{X}_{t-j}(u_0) - \mu(u_0))^2$ .

From (5.7) and (5.8), it follows that

$$\begin{aligned} w_{t_0, N}(\hat{\theta}_{t_0, N}) &= \hat{\alpha}_0\left(\frac{t_0}{N}\right) + \sum_{j=1}^p \hat{\alpha}_j\left(\frac{t_0}{N}\right) \left(X_{t_0-j, N} - \hat{\mu}\left(\frac{t_0}{N}\right)\right)^2 \\ &\xrightarrow{p} \alpha_0(u_0) + \sum_{j=1}^p \alpha_j(u_0) (\tilde{X}_{t_0-j}(u_0) - \mu(u_0))^2 = \sigma_{t_0}(u_0)^2, \end{aligned} \quad (5.9)$$

and

$$\hat{\mu}\left(\frac{t_0}{N}\right) \xrightarrow{p} \mu(u_0). \quad (5.10)$$

Since  $\tilde{\phi}$  is continuous, we obtain

$$\begin{aligned} \log \tilde{\phi}(w_{t_0, N}(\hat{\theta}_{t_0, N})^{\frac{1}{2}}) &= \log \int_{-\infty}^{\infty} \exp\left\{w_{t_0, N}(\hat{\theta}_{t_0, N})^{\frac{1}{2}} x\right\} \phi(x) dx \\ &\xrightarrow{p} \log \int_{-\infty}^{\infty} e^{\sigma_{t_0}(u_0)x} \phi(x) dx. \end{aligned} \quad (5.11)$$

Therefore, from (5.9), (5.10), and (5.11), we can see that

$$\hat{A}_{t_0, N} \equiv \frac{\hat{\mu}_{t_0, N} + \log(\tilde{\phi}(w_{t_0, N}(\hat{\theta}_{t_0, N})^{\frac{1}{2}}))}{w_{t_0, N}(\hat{\theta}_{t_0, N})^{\frac{1}{2}}} \xrightarrow{p} \frac{\mu_{t_0}(u_0) + \log \phi(\sigma_{t_0}(u_0))}{\sigma_{t_0}(u_0)} \equiv A_{t_0}(u_0).$$

□

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